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ON THE LINES OF CERTAIN CLASSES OF SOLENOIDAL OR LAMELLAR VECTORS, SYMMETRICAL WITH RESPECT TO AN AXIS.

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The lines of force due to a homogeneous body bounded by a surface of revolution are curves, each of which lies in a plane passing through the fixed axis of symmetry of the body. The force in such a case is an example of a vector "symmetrical with respect to a straight line," and it is evident that the whole field of a vector of this kind may be studied by examining the lines of the vector in any plane which passes through the axis. If we represent by R, Φ, X the components of a vector taken in the directions in which the columnar coördinates r, ϕ, x increase most rapidly, Φ is everywhere zero if the vector is symmetrical about the x axis.

If $u \equiv f_1(x, y) = k$ is the equation of a family of curves of one parameter in the xy plane, which are lines of a vector symmetrical with respect to the x axis, we may regard $f_1(x, r) = k$, $\phi = m$ as the equations of all the lines of the vector, and the components of the vector satisfy the equations

$$\Phi = 0, \qquad R \cdot \frac{\partial f_1}{\partial r} + X \cdot \frac{\partial f_1}{\partial x} = 0. \tag{1}$$

Given any family of curves, $u = k$, in the xy plane, without multiple points or points of intersection with each other or with the x axis, it is possible to form an infinite number of vectors symmetrical about the x axis which shall have the u curves which lie on one side of it, as lines; for we may choose either R or X wholly at pleasure and determine the other by means of equation (1). It is evident that if two such vectors, $[R_1, 0, X_1]$, $[R_2, 0, X_2]$, have the same lines, $R_1 / R_2 = X_1 / X_2$.

If $v \equiv f_2(x, y) = l$ represents a family of curves in the xy plane orthogonal to the family $u = k$, so that

$$\frac{\partial f_1}{\partial x} \cdot \frac{\partial f_2}{\partial x} + \frac{\partial f_1}{\partial y} \cdot \frac{\partial f_2}{\partial y} = 0, \tag{2}$$

and if $F(v)$ is any single-valued, differentiable function of v, a vector the components of which are

$$R \equiv F'(v) \cdot \frac{\partial f_2(x, r)}{\partial r}, \ \Phi \equiv 0, \ X \equiv F'(v) \cdot \frac{\partial f_2(x, r)}{\partial x} \tag{3}$$

has the u curves as lines, and it is lamellar, for

$$\frac{\partial R}{\partial x} = \frac{\partial X}{\partial r}.$$

Of all the vectors symmetrical about the x axis which have the given u curves as lines, an infinite number are lamellar.

Since the divergence of a vector symmetrical about the x axis is equal to

$$\frac{\partial (r \cdot R)}{r \cdot \partial r} + \frac{\partial X}{\partial x}, \tag{4}$$

it is evident that if $[R, 0, X]$ is a solenoidal vector which has the u curves as lines, $[R \cdot F(u), 0, X \cdot F(u)]$, where F is any single-valued, differentiable function, is another solenoidal vector which has the same lines. If two solenoidal vectors, $[R_1, 0, X_1]$, $[R_2, 0, X_2]$, symmetrical about the x axis have the u curves as lines, the ratio R_1 / R_2, or X_1 / X_2 is a function of u only.

Whatever u is, the vector which has the components

$$-\frac{W}{h_u} \cdot \frac{\partial u}{\partial x}, \ 0, \ +\frac{W}{h_u} \cdot \frac{\partial u}{\partial r}, \tag{5}$$

or the components

$$\frac{W}{h_v} \cdot \frac{\partial v}{\partial r}, \ 0, \ \frac{W}{h_v} \cdot \frac{\partial v}{\partial x}, \tag{6}$$

— where W is a single-valued function of the space coördinates and h_u, h_v are the gradients of the functions u and v, — has the u curves as lines. The tensor of this vector involves u alone or v alone, according as W is expressible in terms of u alone or in terms of v alone.

It is to be remembered that the field of a physical vector may be a restricted region in space, so that a family of u curves which have double points, or points of intersection with each other, or with the x axis, may still be lines of a vector symmetrical with respect to the x axis if the field of the vector is free from such singular points.

If a set of orthogonal curvilinear coördinates in the plane $\phi = 0$ be defined by the functions, $u \equiv f_1(x, r)$, $v \equiv f_2(x, r)$, and if

$$U \equiv \xi(x, r), \; V \equiv \eta(x, r), \; \Phi = 0, \tag{7}$$

represent the magnitudes, at the point (x, r) of the components, taken in the directions in which u, v, ϕ increase most rapidly, of a vector Q, symmetrical with respect to the x axis, it is easy to prove that the divergence of Q is given by the expressions

$$\frac{h_u \cdot h_v}{r} \left\{ \frac{\partial}{\partial u}\left(\frac{r \cdot U}{h_v}\right) + \frac{\partial}{\partial v}\left(\frac{r \cdot V}{h_u}\right) \right\}, \tag{8}$$

$$h_u^2 \cdot \frac{\partial}{\partial u}\left(\frac{U}{h_u}\right) + L(u) \cdot \left(\frac{U}{h_u}\right) + h_v^2 \cdot \frac{\partial}{\partial v}\left(\frac{V}{h_v}\right) + L(v) \cdot \left(\frac{V}{h_v}\right), \tag{9}$$

$$\text{where } L(u) \equiv \frac{\partial^2 u}{\partial r^2} + \frac{\partial u}{r \cdot \partial r} + \frac{\partial^2 u}{\partial x^2} \text{ and } h_u^2 \equiv \left(\frac{\partial u}{\partial r}\right)^2 + \left(\frac{\partial u}{\partial x}\right)^2. \tag{10}$$

The components, (K_u, K_v, K_ϕ) of the curl of Q are

$$0, 0, \text{ and } h_u \cdot h_v \left\{ \frac{\partial}{\partial u}\left(\frac{V}{h_v}\right) - \frac{\partial}{\partial v}\left(\frac{U}{h_u}\right) \right\}. \tag{11}$$

It is to be noticed that

$$\frac{\partial r}{\partial u} = \frac{1}{h_u^2} \cdot \frac{\partial u}{\partial r}, \qquad h_u \cdot \frac{\partial h_u}{\partial r} = \frac{\partial u}{\partial r} \cdot \frac{\partial^2 u}{\partial r^2} + \frac{\partial u}{\partial x} \cdot \frac{\partial^2 u}{\partial r \cdot \partial x},$$

$$h_u^2 \cdot \frac{\partial h_u}{\partial u} = \frac{\partial h_u}{\partial r} \cdot \frac{\partial u}{\partial r} + \frac{\partial h_u}{\partial x} \cdot \frac{\partial u}{\partial x}, \qquad \frac{L(u)}{h_u^2} = \frac{\partial}{\partial u} \log\left(\frac{r \cdot h_u}{h_v}\right),$$

$$\text{and } \frac{L(v)}{h_v^2} = \frac{\partial}{\partial v} \log\left(\frac{r \cdot h_v}{h_u}\right). \tag{12}$$

If the lines of Q in the plane $\phi = 0$ coincide with the u curves, the vector has no component perpendicular to these curves, and U is everywhere zero, so that

$$\text{Divergence } Q = h_v^2 \cdot \frac{\partial}{\partial v}\left(\frac{V}{h_v}\right) + L(v)\left(\frac{V}{h_v}\right), \tag{13}$$

$$= \frac{h_u \cdot h_v}{r} \cdot \frac{\partial}{\partial v}\left(\frac{r V}{h_u}\right), \tag{14}$$

$$\text{Tensor curl } Q = h_u \cdot h_v \cdot \frac{\partial}{\partial u}\left(\frac{V}{h_v}\right). \tag{15}$$

If P_u, P_v, P_ϕ are the components of a vector P taken in the directions in which u, v, ϕ increase most rapidly, the components of the curl of P are

$$\Pi_u = \frac{h_v}{r}\left\{\frac{\partial}{\partial v}(r \cdot P_\phi) - \frac{\partial}{\partial \phi}\left(\frac{P_v}{h_v}\right)\right\}, \tag{16}$$

$$\Pi_v = \frac{h_u}{r}\left\{\frac{\partial}{\partial \phi}\left(\frac{P_u}{h_u}\right) - \frac{\partial}{\partial u}(r \cdot P_\phi)\right\}, \tag{17}$$

$$\Pi_w = h_u \cdot h_v \left\{\frac{\partial}{\partial u}\left(\frac{P_v}{h_v}\right) - \frac{\partial}{\partial v}\left(\frac{P_u}{h_u}\right)\right\}; \tag{18}$$

and, if P is to be a vector potential function of a given solenoidal vector, $Q = [0,\ V,\ 0]$, which has the u curves as lines and is symmetrical about the x axis, we may assume that P_u and P_v involve u and v only, and write

$$P_\phi = \frac{f(u)}{r}, \tag{19}$$

where

$$V = -\frac{h_u}{r} \cdot \frac{\partial}{\partial u} f(u). \tag{20}$$

Any vector of the form $[P_u, P_v, f(u)/r]$, where P_u and P_v are any functions of u and v subject only to the condition

$$\frac{\partial}{\partial u}\left(\frac{P_v}{h_v}\right) = \frac{\partial}{\partial v}\left(\frac{P_u}{h_u}\right), \tag{21}$$

is a vector potential function of a solenoidal vector, symmetrical about the x axis, which has the u curves for lines; and there is no vector of the kind last mentioned which does not have as a potential function a vector P of the form given. It is usually convenient to make $P_u = P_v = 0$.

It is easy to see from the foregoing equations that the statements which follow are true.

(*a*) If Q is to be solenoidal, $r \cdot V/h_u$ must be either constant or a function of u only; that is,

$$V = \frac{h_u \cdot F(u)}{r}, \tag{22}$$

where $F(u)$ is some single-valued function: if V is of this form it is solenoidal. It follows from this that of all the vectors symmetrical about the x axis which have the u curves as lines an infinite number are solenoidal. If the u curves are straight lines parallel to the x axis, the tensor of Q is some function of r or else constant.

(*b*) If Q is to be solenoidal and if its tensor is to be either constant or expressible in terms of u, the gradient of the function u must satisfy an equation of the form

$$h_u = r \cdot F(u). \tag{23}$$

If for the function u in this equation we substitute a new function w defined by the equation

$$w = \int \frac{du}{F(u)},$$

we shall get the simpler equation

$$h_w = r. \tag{24}$$

It is to be noticed that w is constant on any line of constant u, and that (23) and (24) may be said to define the same curves.

(*c*) If Q is to be solenoidal and if its tensor is to be expressible in terms of v only, h_u must be of the form

$$h_u = r \cdot F(u) \cdot \psi(v), \tag{25}$$

and if for u we substitute w in the manner indicated in (*b*) we shall obtain the equation

$$h_w = r \cdot \psi(v), \tag{26}$$

or

$$\frac{\partial}{\partial w}\left(\frac{h_w}{r}\right) = 0. \tag{27}$$

(*d*) If a solenoidal vector symmetrical about the x axis has the u curves for lines, its curl is of the form

$$\left[0,\ 0,\ h_u \cdot h_v \cdot \frac{\partial}{\partial u}\left(\frac{h_u \cdot F(u)}{r \cdot h_v}\right)\right] \quad \text{or}$$

$$\left[0,\ 0,\ r \cdot h_u^2 \cdot \frac{\partial}{\partial u}\left(\frac{F(u)}{r^2}\right) + \frac{L(u)}{r} \cdot F(u)\right]. \tag{28}$$

If, for instance, the lines of such a vector are straight lines parallel to the x axis, its curl is either constant or a function of r alone.

(*e*) If Q is to be lamellar, it must be of the form

$$V = h_v \cdot F(v). \tag{29}$$

Any vector of the form $[0, h_v \cdot F(v), 0]$ is lamellar.

(*f*) If Q is to be lamellar and if its tensor is to involve v only, we must have

$$h_v = \psi(v), \tag{30}$$

and the v curves must form a system of parallel, curved or straight, lines (for instance, a set of concentric circumferences); the u curves are, therefore, straight lines.

If for v in (30) we substitute a new function z such that

$$z \equiv \int \frac{dv}{\psi(v)},$$

we shall get the new equation

$$h_z = 1. \tag{31}$$

(*g*) If Q is to be lamellar while its tensor involves u only, we must have

$$h_v = F(u) \cdot \psi(v), \tag{32}$$

and the substitution used in (*f*) leads to the condition

$$h_z = F(u). \tag{33}$$

We may consider that (32) and (33) define the same systems of curves. By making use of the so-called "Principle of Duality," and putting

$$m = -2 \log (x^2 + r^2), \quad n = \tan^{-1}(r/x), \tag{34}$$

it is possible* to reduce (33) to the equivalent of Fourier's equation for the linear flow of heat.

If the v curves were a family of straight lines emanating from some fixed point (x_0, r_0) the equations of these lines might be written in the form

$$v = \frac{r - r_0}{x - x_0},$$

and the equation of the orthogonal curves in the form

$$u^2 = (x - x_0)^2 + (r - r_0)^2:$$

in this case we should have

$$h_v = (1 + v^2)/u,$$

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* Peirce, These Proceedings, **38**, p. 663.

so that (32) would be satisfied. If the equation of the same family of straight lines had been written in the form

$$v = \tan^{-1}\left(\frac{r - r_0}{x - x_0}\right)$$

we should have had

$$h_v = 1 / u,$$

so that v would have satisfied an equation of the form (33).

(*h*) If Q is lamellar, and if Ω is a scalar potential function of Q, Ω must be expressible in terms of v, and the divergence of Q is equal to

$$h_v^2 \cdot \frac{d^2\Omega}{dv^2} + L(v) \cdot \frac{d\Omega}{dv}, \tag{35}$$

(*i*) If the tensor of Q has the same value at all points of the $x\,r$ plane Q is lamellar if, and only if, h_v is constant or a function of v alone.

(*j*) If the tensor of a vector, Q, which has the u curves as lines, is a function of u only, its divergence is

$$\frac{h_u \cdot h_v \cdot V}{r} \frac{\partial}{\partial v}\left(\frac{r}{h_u}\right). \tag{36}$$

(*k*) If the tensor of Q is expressible in terms of v, the tensor of its curl is

$$h_u \cdot h_v \cdot V \cdot \frac{\partial}{\partial u}\left(\frac{1}{h_v}\right). \tag{37}$$

If, for instance, the u curves are a family of straight lines, the tensor of the curl of such a vector must be zero.

(*l*) If Q is to be solenoidal as well as lamellar, equations (13) and (15) yield Lamé's well-known condition that $L(v)/h_v^2$ must be expressible in terms of v alone, so that

$$\frac{\partial}{\partial u}\left(\frac{L(v)}{h_v^2}\right) = 0. \tag{38}$$

(*m*) A vector symmetrical about the x axis and directed everywhere parallel to it, is solenoidal only when its tensor has the same value throughout every one of its lines.

Systems of Straight Lines and Circles in the xr Plane which satisfy Lamé's Condition.

If a set of curves ($u = k$) in the rx plane yield when revolved about the x axis a set of isothermal * surfaces, the function

$$\frac{L(u)}{h_u^2} \tag{39}$$

must be expressible as a function of u alone. The families of curves which satisfy this condition are generally, of course, quite different from those which satisfy the condition

$$\frac{\nabla^2(u)}{h_u^2}, \text{ a function of } u, \tag{40}$$

for isothermal lines in the plane. A set of confocal conics with foci on the x axis would, however, satisfy † both conditions.

To determine what systems of straight lines in the xy plane satisfy (39) we may represent any such system by the equation $\alpha x + \beta y = 1$, where α and β are functions of a single parameter u, and write

$$\alpha' \equiv \frac{d\alpha}{du}, \qquad \beta' \equiv \frac{d\beta}{du}.$$

$$\frac{\partial u}{\partial x} = \frac{-\alpha}{\alpha' x + \beta' y}, \qquad \frac{\partial u}{\partial y} = \frac{-\beta}{\alpha' x + \beta' y}, \qquad h_u^2 = \frac{\alpha^2 + \beta^2}{(\alpha' x + \beta' y)^2},$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{2\alpha\alpha'}{(\alpha' x + \beta' y)^2} - \frac{\alpha^2(\alpha'' x + \beta'' y)}{(\alpha' x + \beta' y)^3},$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{2\beta\beta'}{(\alpha' x + \beta' y)^2} - \frac{\beta^2(\alpha'' x + \beta'' y)}{(\alpha' x + \beta' y)^3},$$

$$\frac{L(u)}{h_u^2} = \frac{2(\alpha\alpha' + \beta\beta')}{\alpha^2 + \beta^2} - \frac{\alpha'' x + \beta'' y}{\alpha' x + \beta' y} - \frac{\alpha(\alpha' x + \beta' y)}{y(\alpha^2 + \beta^2)}. \tag{41}$$

In order that the sum of the last two terms of (41) may be expressible in terms of u only, it is necessary that α' shall be zero, so that α is only a constant, and the equation of any system of straight lines which satisfy (39) is of the form

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* Lamé, Leçons sur les coordonnées curvilignes, p. 32; Leçons sur les fonctions inverses, p. 5; Somoff-Ziwet, Theoretische Mechanik, I. 113 and 138.

† Peirce, American Journal of Mathematics, 1896.

$$\frac{1 - c\,x}{y} = \text{parameter.}$$

All the lines of such a system pass through a fixed point $(1/c,\ 0)$ on the x axis.

Again: let

$$x^2 + y^2 - 2\,\alpha\,x - 2\,\beta\,y - \gamma = 0 \tag{42}$$

where α, β, γ are functions of a single parameter u, represent a family of circles in the $x\,y$ plane so that

$$\frac{\partial u}{\partial x} = \frac{2\,(x - \alpha)}{2\,\alpha' x + 2\,\beta' y + \gamma'}, \qquad \frac{\partial u}{\partial y} = \frac{2\,(y - \beta)}{2\,\alpha' x + 2\,\beta' y + \gamma'},$$

$$h_u^{\,2} = \frac{4\,(\alpha^2 + \beta^2 + \gamma)}{(2\,\alpha' x + 2\,\beta' y + \gamma')^2}.$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{2}{2\,\alpha' x + 2\,\beta' y + \gamma'} - \frac{8\,\alpha'\,(x - \alpha)}{(2\,\alpha' x + 2\,\beta' y + \gamma')^2} - \frac{4\,(x - \alpha)^2\,(2\,\alpha'' x + 2\,\beta'' y + \gamma'')}{(2\,\alpha' x + 2\,\beta' y + \gamma')^3}.$$

$$\frac{L\,(u)}{h_u^{\,2}} = \frac{2\,\alpha\,\alpha' + 2\,\beta\,\beta' + \gamma'}{\alpha^2 + \beta^2 + \gamma} - \frac{2\,\alpha'' x + 2\,\beta'' y + \gamma''}{2\,\alpha' x + 2\,\beta' y + \gamma'} + \frac{(y - \beta)\,(2\,\alpha' x + 2\,\beta' y + \gamma')}{4\,y\,(\alpha^2 + \beta^2 + \gamma)}. \tag{43}$$

The first term in the second number of (43) is already expressed as a function of u: the sum of the last two terms is a function of u if, and only if, $\alpha' = 0$, $\beta = 0$, so that α is a constant (c) and (42) takes the form

$$(x - a)^2 + y^2 = u. \tag{44}$$

This represents a set of concentric circles with centre at same point on the x axis.

THE EQUATION $h_u = y$.

If the tensor of a solenoidal vector, symmetrical about the x axis, is to have at every point of each of its lines a value constant for that line, the tensor is a function of the parameter of the lines and the equation of the family must be found among the solutions of equation (24).

If u is a differentiable function of x and y, the partial derivations of which with respect to these variables are p and q respectively, we may use the usual notation and rewrite (24) in the form

$$p^2 = y^2 - q^2. \tag{45}$$

Equating each side* to a constant, (a^2), we get

$$u = \pm\, a x + \tau(y) = \tfrac{1}{2} y \sqrt{y^2 - a^2} - \tfrac{1}{2} a^2 \,.\, \log\left(y + \sqrt{y^2 - a^2}\right) + \chi(x),$$

so that the complete integral of (45) is

$$u = \pm\, a x + \tfrac{1}{2} y \sqrt{y^2 - a^2} - \tfrac{1}{2} a^2 \,.\, \log\left(y + \sqrt{y^2 - a^2}\right) + b, \tag{46}$$

and the general integral can be formed from this directly.

A special solution gives straight lines parallel to the x axis as possible lines of a solenoidal symmetrical vector the tensor of which is a function only of the distance from the axis of symmetry. No solenoidal symmetrical vector the tensor of which involved x only could have these lines.

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* Forsyth, Differential Equations, p. 310.